Lab 2: Time Series

Econometrics Beyond Ordinary Least Squares

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Time Series

Time Series: Consecutive **realizations** of the same **random variable** Y (eg. production output) in periods $t \in T = (1, 2, ..., T)$.

Auto-Regression (AR): Meaningful impact of past periods' realizations on current realization.

$$Y_t = \delta + \theta Y_{t-1} + \epsilon_t \tag{1}$$

$$E[\epsilon_t] = 0 \forall t \in T;\ Var[\epsilon_t] = \sigma_\epsilon^2 \forall t \in T;\ Cov[\epsilon_t, \epsilon_s] = 0 \forall s \neq t \in T;\ \mid \theta \mid < 1$$

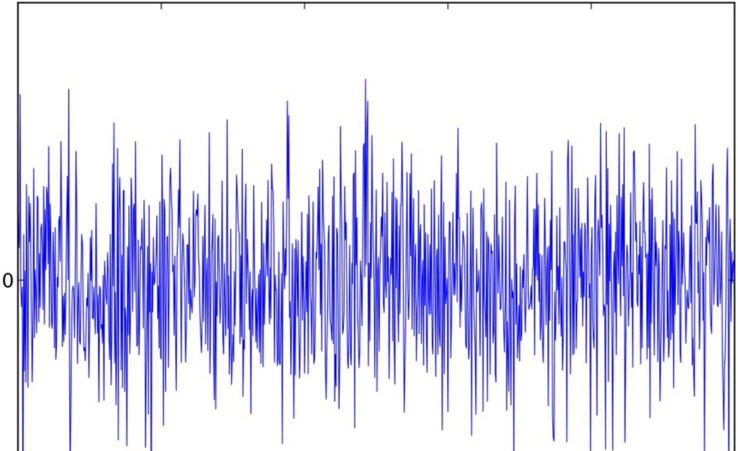
Moving Average (MA): Past periods' deviations from predicted values impact current realization.

$$Y_t = \mu + \epsilon_t + \alpha_1 \epsilon_{t-1} \tag{2}$$

White Noise

ϵ_t will usually be assumed to be **white noise**:

- serially uncorrelated
- 0 mean
- finite variance



Autoregression 1

$$Y_t = \delta + \theta Y_{t-1} + \epsilon_t$$

Expected Value:

$$\begin{aligned} E[Y_t] &= \delta + \theta E[Y_{t-1}] \\ E[Y_t \mid t] &= E[Y_t] = E[Y_{t-1}] \text{ (expected value does not depend on t)} \\ \mu = E[Y_t] &= \frac{\delta}{1-\theta} \end{aligned} \tag{3}$$

Define $y_t = Y_t - \mu$, lose the intercept term δ.

Autoregression 2

Variance:

$$V[Y_t] = V[\delta + \theta Y_{t-1} + \epsilon_t] = V[\theta Y_{t-1}\epsilon_t] = \theta^2 V[Y_{t-1}] + V[\epsilon_t]$$

$$V[Y_t] = V[Y_{t-1}] \Rightarrow V[Y_t] = \frac{\sigma^2}{1-\theta} \quad (4)$$

Covariance:

$$Cov(y_t, y_{t-1}) = E[y_t, y_{t-1}] = E[(\theta y_{t-1} + \epsilon_t)y_{t-1}] = \theta V[y_{t-1}] = \theta \frac{\sigma^2}{1-\theta^2} \quad (5)$$

$$Cov(y_t, y_{t-k}) = \theta^k \frac{\sigma^2}{1-\theta^2} \quad (6)$$

Moving Averages 1

$$Y_t = \mu + \epsilon_t + \alpha\epsilon_{t-1}$$
$$E[Y_t] = \mu \quad (7)$$

Variance and Covariance:

$$V[Y_t] = [(\epsilon_t + \alpha\epsilon_{t-1})^2] = E[\epsilon_t^2] + \alpha^2 E[\epsilon_{t-1}^2] = (1+\alpha^2)\sigma^2 \quad (8)$$
$$Cov[Y_t, Y_{t-1}] = E[(\epsilon_t + \alpha\epsilon_{t-1})(\epsilon_{t-1} + \alpha\epsilon_{t-2}] = \alpha E[\epsilon_{t-1}^2] = \alpha\sigma^2 \quad (9)$$
$$Cov[Y_t, Y_{t-2}] = E[(\epsilon_t + \alpha\epsilon_{t-1})(\epsilon_{t-2} + \alpha\epsilon_{t-3}] = 0 \quad (10)$$

Relationship between AR and MA

- AR is a "**long-memory**" process, MA has an auto-covariance of 0 for all distances greater than 1 period.
- AR can be written as an infinite-order MA process if $\mid \theta \mid < 1$:

$$
\begin{aligned}
Y_t &= \delta + \theta Y_{t-1} + \epsilon_t \\
Y_{t-1} &= \delta + \theta Y_{t-2} + \epsilon_{t-1} \\
\Rightarrow Y_t &= \mu + \theta^2 (Y_{t-2} - \mu) + \epsilon_t + \theta \epsilon_{t-1} \\
\Rightarrow Y_t &= \theta^n (Y_{t-2} - \mu) + \sum_{j=0}^{N-1} \theta^j \epsilon_{t-j} \\
\lim_{n \to \infty} Y_t &= \mu + \sum_{j=0}^{\infty} \theta^j \epsilon_{t-j} \qquad (11)
\end{aligned}
$$

Autocovariance and Autocorrelation Functions (ACF)

$$\gamma_k = Cov[Y_t, Y_{t-k}] = Cov[Y_{t-k}, Y_t] \text{ (Autocovariance)} \tag{12}$$

$$\rho_k = \frac{Cov[Y_t, Y_{t-k}]}{V[Y_t]} = \frac{\gamma_k}{\gamma_0} \text{ Autocorrellation} \tag{13}$$

$$\gamma_k \in [-\infty, \infty] \; ; \; \rho_k \in [-1, 1]$$

Autocorrelation in AR

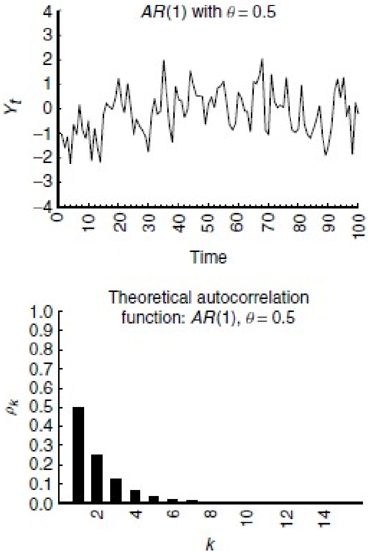


Figure 2: Verbeek, 2003, Figure 8.1: “First order autoregressive processes: data series and autocorrelation functions”

Autocorrelation in MA

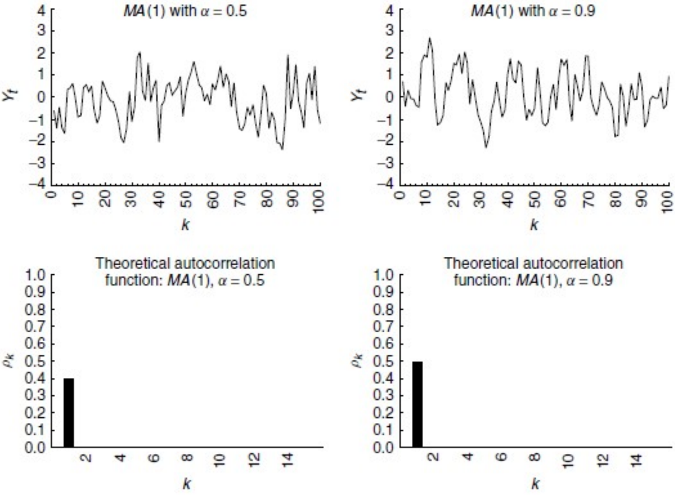


Figure 8.2 First order moving average processes: data series and autocorrelation functions

Figure 3: Verbeek, 2003, Figure 8.2: "First order moving average processes: data series and autocorrelation functions"

Stationarity

Stationarity is a statistical concept describing **joint distributions**.

- Engineers have it easier: "Stationarity can be defined in precise mathematical terms, but for our purpose we mean a flat looking series [. . .]" (NIST/SEMATECH e-Handbook of Statistical Methods, https://www.itl.nist.gov/div898/handbook/pmc/section4/pmc442.htm)
- Frequentist Econometricians have it harder: We pretend to attempt and find evidence **against** our hypothesis, then are relieved when we don't find it.
- Essentially stationarity/unit root testing is defining how a distribution could look if it was **not stationary**, then compare our data to this.

Strict Stationarity: Properties of a process are unaffected by a change in its time origin.

Weak/Covariance Stationarity: Mean, Variance and Covariance of a joint distribution are unaffected by a change of time origin.

Econometric statements usually concern **distributional moments** (eg. change in $E[Y \mid X]$), if these change over time $\Rightarrow$ loss of generality.

Weak Stationarity

$$E[Y_t] = \mu \quad \forall t \in T \tag{14}$$

$$V[Y_t] = \sigma^2 = \gamma_0 < \infty \quad \forall t \in T \tag{15}$$

$$Cov[Y_t, Y_{t-s}] = E[(Y_t - \mu)(Y_{t-k} - \mu)] = \gamma_k \quad \forall k < t \tag{16}$$

ARMA processes

A Moving Average process of **order q**:

$$y_t = \epsilon_t + \alpha_1 \epsilon_{t-1} + ... + \alpha_q \epsilon_{t-q} \tag{17}$$

An Autoregressive process of **order p**:

$$y_t = \theta_1 y_{t-1} + ... + \theta_p y_{t-p} + \epsilon_t \tag{18}$$

An **ARMA(p, q)** process:

$$y_t = \theta_1 y_{t-1} + ... + \theta_p y_{t-p} + \epsilon_t + \alpha_1 \epsilon_{t-1} + ... + \alpha_q \epsilon_{t-q} \tag{19}$$

Lag Operators

$$Ly_t = y_{t-1} \tag{20}$$
$$L^2 y_t = L(Ly_t) = y_{t-2}$$

AR(1) in lag notation:

$$y_t = \theta y_{t-1} + \epsilon_t = \theta L y_t + \epsilon_t$$
$$\epsilon_t = (1 - \theta L) y_t \tag{21}$$

AR(p) in **lag polynomial** notation with lag polynomial $\theta(L)$. The lag polynomial is a filter, when applied to an AR(p) process it produces a **white noise process** ϵ.

$$\theta(L) y_t = \epsilon_t$$
$$\theta(L) = 1 - \theta_1 L - \theta_2 L^2 - \ldots - \theta_p L^p \tag{22}$$

Inverse Lag Polynomial

Inverse lag polynomial $\theta^{-1}L : \theta^{-1}(L)\theta(L) = 1$. An AR polynomial is **invertible** if $\mid \theta \mid < 1$.

$$
\begin{aligned}
(1 - \theta L)y_t &= \epsilon_t \\
(1 - \theta L)^{-1} &= \sum_{j=0}^{\infty} \theta^j L^j
\end{aligned} \tag{23}
$$

Inverse lag polynomials allow to re-write MA processes in **finite order AR terms**. The MA polynomial is invertible if $\mid \alpha \mid < 1$.

$$
\begin{aligned}
y_t &= \epsilon_t + \alpha_1 \epsilon_{t-1} + ... + \alpha_q \epsilon_{t-q} \\
y_t &= \alpha(L)\epsilon_t \\
\alpha(L) &= 1 + \alpha_q L + \alpha_2 L^2 + ... + \alpha_q L^q
\end{aligned} \tag{24}
$$

$$
\alpha^{-1}(L)y_t = \epsilon_t \text{ An AR(q=}\infty\text{) process} \tag{25}
$$

Characteristic Roots

Rewrite the second order lag polynomial with roots $\phi = (\phi_1, \phi_2)$.

$$1 - \theta_1 L - \theta_2 L^2 = (1 - \phi_1 L)(1 - \phi_2 L) \tag{26}$$

The polynomial is invertible if both $1 - \phi_1 L$ and $1 - \phi_2 L$ are invertible, ie. $\mid \phi_1 \mid < 1$ and $\mid \phi_1 \mid < 1$.

This can be tested by formulating the **characteristic equation**. It can be solved for two roots z_1, z_2. If any root is smaller or equal 1, it is called a **unit root** and the polynomial is **not invertible**.

$$(1 - \phi 1 z)(1 - \phi_2 z) = 0 \tag{27}$$

Unit Roots Example

$$\begin{aligned} y_t &= 1.2y_{t-1} - 0.32y_{t-2} + \epsilon_t \\ \epsilon_t &= (1-0.8L)(1-0.4L)y_t \\ 1 - 1.2z + 0.32z^2 &= (1-0.8z)(1-0.4z) = 0 \\ z_1 &= \frac{1}{0.8} > 1 \text{ ; } z_2 = \frac{1}{0.4} > 1 \qquad (28) \end{aligned}$$

$\Rightarrow$ The AR polynomial is invertible.

Unit Roots and Stationarity

Any finite order MA process is stationary, because it is the **weighted sum of stationary white noise** processes ϵ_t by design.

An auto-regressive process with $\theta \geq 1$ is not stationary, because its variance cannot be solved analytically (unless $\sigma^2 = V[\epsilon] = 0$).

$$y_t = \theta y_{t-1} + \epsilon_t \text{ s.t. } \theta = 1$$
$$V[y_t] = V[y_{t-1}] + V[\epsilon_t] = V[y_t] + \sigma^2$$

More precisely, an AR process with $\theta = 1$ is called a **random walk** (a process for which $E[Y_t] = E[Y_{t-1}]$).

With $\theta > 1$ it is **non-stationary**.

Unit Root Testing: Dickey and Fuller

The Dickey-Fuller and Augmented Dickey-Fuller test construct a simple test statistic DF and provide tables of critical values to reject $H_0 :\mid \theta \mid = 1$.

$$\begin{aligned} Y_t &= \delta + \theta Y_{t-1} + \epsilon_t \\ DF_\mu &= \frac{\hat{\theta} - 1}{s.e.(\hat{\theta})} \end{aligned} \tag{29}$$

Augmented Dickey Fuller test regression:

$$\begin{aligned} \Delta Y_t &= \delta + \gamma Y_{t-1} + \epsilon_t \\ DF_\tau &= \frac{\hat{\gamma}}{s.e.(\hat{\gamma})} \end{aligned} \tag{30}$$